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# A SURVEY OF SOME SECOND-ORDER DIFFERENCE SCHEMES FOR THE STEADY-STATE CONVECTION–DIFFUSION EQUATION

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SUMMARY

This paper presents a survey of several finite difference schemes for the steady-state convection–diffusion equation in one and two dimensions. Most difference schemes have $O(h^2)$ truncation error. The behaviour of these schemes on a one-dimensional model problem is analysed in detail, especially for the case when convection dominates diffusion. It is concluded that none of these schemes is universally second order. One recently proposed scheme is found to yield highly inaccurate solutions for the case of practical interest, i.e. when convection dominates diffusion. Extensions to two and three dimensions are also discussed.

KEY WORDS Convection Diffusion Equation Linearized Burger's Equation Finite Difference Schemes Accuracy Convection Dominated Flows

## 1. INTRODUCTION

In this paper we consider the steady-state convection–diffusion equation

$$Lu \equiv u_{xx} + u_{yy} + \lambda_1 u_x + \lambda_2 u_y = f(x, y) \tag{1}$$

where $\lambda_1, \lambda_2$ are constants that may take large values. Equation (1) holds in a bounded domain $D$ with boundary $\Gamma$. The values of $u(x, y)$ on the boundary of $D$ are assumed known.

The above differential equation is a linearized version of the differential equations that describe the steady transport of momentum, energy, vorticity, etc. The problems of physical interest and practical importance include those where the convection (advection) dominates diffusion. These problems correspond to equation (1) with large values of $\lambda_1, \lambda_2$.

The above differential equation has been studied by a large number of investigators and many finite difference schemes have been proposed in the literature. The main object of these investigations has been to find a difference scheme that has a high order of truncation error and yields accurate solutions when $\lambda_i$ are large.

In this paper we examine six difference schemes, five of which have truncation errors of order $h^2$, in the limiting case when $\lambda_i$ are fixed and the mesh width $h$ is reduced. These schemes include one going as far back as Allen and Southwell[2] and a recent one proposed by Dennis *et al.*[4] The central difference scheme and the upwind scheme of first order are also included. We examine the behaviour of these schemes for the case of practical importance: $h$ fixed and $\lambda_i$ moderately large. In the case of the one-dimensional convection diffusion equation we establish, both analytically and numerically, that all of these second-order

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schemes exhibit $O(h)$ behaviour at best and yield grossly inaccurate solutions at worst. In the case of the two-dimensional problem of equation (1) we establish corresponding results numerically.

In the next two sections we introduce the finite difference schemes and exhibit their truncation errors. In Sections 4 and 5 we closely examine a model problem and analyse the behaviour of each of the difference schemes for two cases: (i) when $h \to 0$ and (ii) when $h$ is fixed and convection dominates diffusion. In Section 6 we present numerical results to support the analysis of Section 5. In Section 7 we present some numerical results for equation (1) and show that the behaviour of these schemes is carried over from one to two dimensions.

It is concluded that none of the schemes considered here for the one-dimensional problem is universally second order. When convection dominates diffusion, one observes either the $O(h)$ behaviour of the upwind scheme or the oscillatory behaviour of the central scheme or the smooth but grossly inaccurate behaviour of the Dennis scheme. It is possible to devise high-order schemes that work well for the convection dominated flows, see e.g. Reference 14. However, such schemes may not exhibit high orders of accuracy for the whole range of $\lambda_j$.

In higher dimensions it is possible to construct schemes that yield stable and accurate solutions for the whole range of $\lambda_j$. Details of numerical experiments with one such scheme are presented in this paper.

## 2. FINITE DIFFERENCE SCHEMES

In this section we consider the one-dimensional analogue of equation (1):

$$\begin{aligned} Lu \equiv u_{xx} + \lambda u_x &= f(x), \qquad a \le x \le b; \\ u(a) = \alpha, &\qquad u(b) = \beta. \end{aligned} \tag{2}$$

Equation (2) is also referred to as the linearized Burger's equation.

We cover the interval $[a, b]$ by a uniform mesh: $\{x_i : x_i = a + ih,\ h = (b-a)/N\}$ and use the notation $u_i = u(x_i)$. The boundary conditions reduce to $u_0 = \alpha$, $u_N = \beta$. At each interior point $x_i$, $1 \le i \le N-1$, we define a finite difference approximation to the operator $Lu$. The following approximations are frequently used in the literature:

1. Upwind Difference Scheme

$$\begin{aligned} L_h^1 u_i &= h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_{i+1} - u_i), \qquad \lambda > 0 \\ &= h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_i - u_{i-1}), \qquad \lambda < 0. \end{aligned}$$

2. Central Difference Scheme

$$L_h^2 u_i = h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda (2h)^{-1}(u_{i+1} - u_{i-1}).$$

3. Il'in Scheme

$$L_h^3 u_i = \frac{\lambda h}{2} \coth\left(\frac{\lambda h}{2}\right) h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda (2h)^{-1}(u_{i+1} - u_{i-1}).$$

4. Samarskii Scheme

$$\begin{aligned} L_h^4 u_i &= (1 + \lambda h/2)^{-1} h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_{i+1} - u_i), \qquad \lambda > 0 \\ &= (1 - \lambda h/2)^{-1} h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_i - u_{i-1}), \qquad \lambda < 0. \end{aligned}$$

5. Dennis Scheme

$$L_h^5 u_i = \left(1 + \frac{\lambda^2 h^2}{8}\right) h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda (2h)^{-1}(u_{i+1} - u_{i-1}).$$

The first two schemes defined by $L_h^1$ and $L_h^2$ have been widely used in the literature. The Il'in scheme $L_h^3$ has been studied by Kellogg *et al.*[10,11] Gresho and Lee[5] refer to this scheme as the 'smart' upwind scheme as it produces the exact solution of equation (2) with $f = 0$. With $f \neq 0$, $L_h^3$ has a discretization error of order $h^2/(h + \lambda^{-1})$.[10] The history of this scheme goes back to Allen and Southwell,[2] Il'in,[9] Roscoe[13] and others. Dennis[3] has an exponential scheme which is very similar to the Il'in scheme. The Samarskii scheme $L_h^4$ has been considered by a few authors.[8,10] The Dennis scheme $L_h^5$ is the one-dimensional analogue of the scheme proposed by Dennis *et al.*[4] for solving a three-dimensional cavity flow problem. This scheme can be obtained from the Il'in scheme $L_h^3$ or from the Dennis exponential scheme[3] by retaining the first three terms (up to order $\theta^2$) in the expansion of $e^{\theta}$ and $e^{-\theta}$.

The upwind scheme $L_h^1$ uses second-order discretization of the diffusion term and first-order one-sided discretization of the convection term. In the central scheme $L_h^2$ each term is replaced by second-order central differences. The Il'in scheme $L_h^3$ and the Dennis scheme $L_h^5$ use a central difference approximation of the convection term; in addition, the diffusion term is multiplied by terms which are of order 1 in the limiting case of $\lambda h \to 0$. The Samarskii scheme $L_h^4$ uses one-sided differences for the approximation of the convection term. The multiplication factor of the diffusion term yields an effective central difference approximation of the convection term when $\lambda h \to 0$.

We also consider a modification of the Samarskii scheme $L_h^4$ that is obtained by retaining the first three terms in the expansion of $(1 \pm \theta)^{-1}$:

$$L_h^6 u_i = \left(1 - \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{4}\right) h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_{i+1} - u_i), \qquad \lambda > 0$$

$$= \left(1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{4}\right) h^{-2}(u_{i+1} - 2u_i + u_{i-1}) + \lambda h^{-1}(u_i - u_{i-1}), \qquad \lambda < 0.$$

## 3. TRUNCATION ERRORS

The truncation errors of the difference schemes $(L_h u - Lu,\ h \to 0)$, assuming sufficient regularity of $u(x)$, are given by:

1. Upwind $\quad \frac{|\lambda|\, h}{2} u_{xx} + h^2/12(u_{xxxx} + 2\lambda u_{xxx}) + O(h^3)$
2. Central $\quad h^2/12(u_{xxxx} + 2\lambda u_{xxx}) + O(h^4)$
3. Il'in $\quad h^2/12(u_{xxxx} + 2\lambda u_{xxx} + \lambda^2 u_{xx}) + O(h^4)$
4. Samarskii $\quad h^2/12(u_{xxxx} + 2\lambda u_{xxx} + 3\lambda^2 u_{xx}) - \frac{|\lambda|^3 h^3}{8} u_{xx} + O(h^4)$
5. Dennis $\quad h^2/12(u_{xxxx} + 2\lambda u_{xxx} + \frac{3}{2}\lambda^2 u_{xx}) + O(h^4)$
6. Modified Samarskii $\quad h^2/12(u_{xxxx} + 2\lambda u_{xxx} + 3\lambda^2 u_{xx}) + O(h^4)$

In each case we have assumed that the value of $\lambda$ remains fixed and the mesh width $h$ is allowed to decrease. Except for the upwind scheme, which has a truncation error of order $h$, all of the other schemes are of order $h^2$.

## 4. MODEL PROBLEM

We first examine the effect of the above difference schemes on the model problem:

$$u'' - \text{Pe} \,.\, u' = 0, \qquad 0 < x < 1; \quad \text{Pe} > 0$$
$$u(0) = T_0, \qquad u(1) = T_1. \tag{3}$$

The exact solution of this problem is given by

$$u(x) = T_0 + (T_1 - T_0) \,.\, \frac{1 - e^{\text{Pe}.x}}{1 - e^{\text{Pe}}} \tag{4}$$

where Pe is the Peclet number. This problem has been studied by many authors, a recent study being Reference 5. The solution $u(x)$ has a smooth variation over the interval (0, 1) when Pe is small. When Pe is large, the solution exhibits a boundary layer behaviour where $u(x)$ is almost equal to $T_0$ except for a thin layer near $x = 1$ in which the solution $u(x)$ rapidly changes from $T_0$ to $T_1$. This boundary layer has thickness $\delta \cong 1/\text{Pe}$ and has been referred to as the Outflow Boundary Layer in Reference 5. In Figure 1 we present the graph of $u(x)$ for several values of Pe.

The finite difference approximations discussed in the previous sections can be rewritten for equation (3) to yield the following linear difference equations of second order. We use the notation $\theta = \text{Pe} \,.\, h/2$, $x_i = ih$, and $h = 1/N$. Each of the following equations is defined for

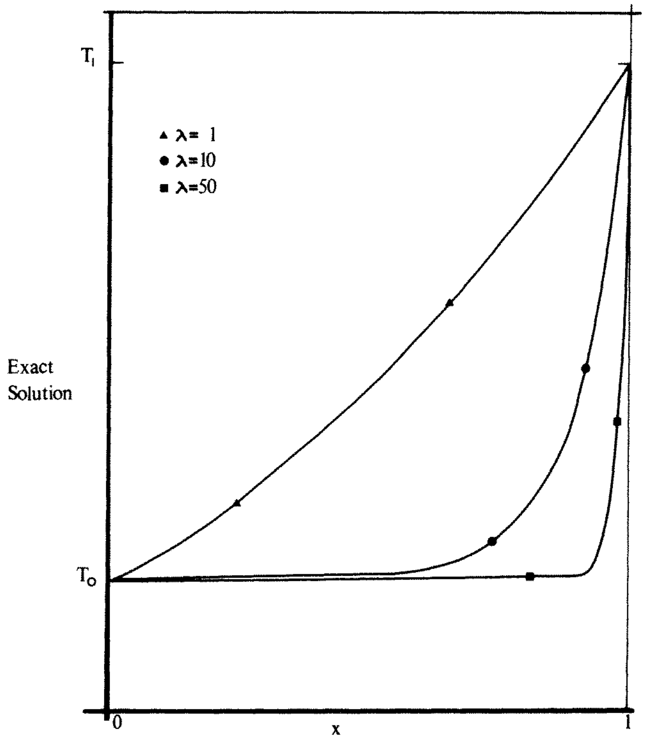


Figure 1. Exact solutions of the one-dimensional model problem ($\lambda$ = Peclet number, $f(x) = 0$)

$i = 1, 2, \ldots, N-1.$

$$h^2 L_h^1 u_i = u_{i+1} - (2+2\theta)u_i + (1+2\theta)u_{i-1} = 0$$

$$h^2 L_h^2 u_i = (1-\theta)u_{i+1} - 2u_i + (1+\theta)u_{i-1} = 0$$

$$h^2 L_h^3 u_i = \frac{\theta}{\sin h\theta}[\mathrm{e}^{-\theta}u_{i+1} - 2\cosh\theta u_i + e^{\theta}u_{i-1}] = 0$$

$$h^2 L_h^4 u_i = \frac{1}{1+\theta}[u_{i+1} - 2(1+\theta+\theta^2)u_i + (1+2\theta+2\theta^2)u_{i-1}] = 0$$

$$h^2 L_h^5 u_i = (1-\theta+\theta^2/2)u_{i+1} - (2+\theta^2)u_i + (1+\theta+\theta^2/2)u_{i-1}] = 0$$

$$h^2 L_h^6 u_i = (1-\theta+\theta^2)u_{i+1} - (2+2\theta^2)u_i + (1+\theta+\theta^2)u_{i-1} = 0$$

Each of the above difference schemes is of the form:

$$K[au_{i+1} - (a+b)u_i + bu_{i-1}] = 0, \qquad 1 \le i \le N-1 \tag{5}$$

The boundary values are $u_0 = T_0$, $u_N = T_1$.

The general solution of equation (5) is given by

$$u_i = c_1 + c_2 z^i \quad \text{where} \quad z = b/a.$$

Using the boundary values, we obtain

$$u_i = T_0 + (T_1 - T_0) \cdot \frac{(1-z^i)}{(1-z^N)}, \qquad 1 \le i \le N-1 \tag{6}$$

The error at $x = x_i$ is given by

$$e_i = u_i - u(x_i) = (T_1 - T_0)\left[\frac{1-z^i}{1-z^N} - \frac{1-\mathrm{e}^{\mathrm{Pe}.x_i}}{1-\mathrm{e}^{\mathrm{Pe}}}\right] \tag{7}$$

We are interested in the behaviour of $e_i$ when

(i) Pe is fixed and $h \to 0$; and more importantly,

(ii) $h$ is fixed and Pe is increased.

## 5. THE MODEL PROBLEM: BEHAVIOUR OF THE APPROXIMATE SOLUTIONS OF THE DIFFERENCE SCHEMES

We first consider the problem of convergence when Pe is fixed and the mesh width $h$ is reduced. The error at $x = x_i$ is defined by equation (7), where $z$ is ratio of the coefficient of $u_{i-1}$ to the coefficient of $u_{i+1}$. The results are summarized in Table I. It is clear that the upwind scheme converges with $O(h)$ error, the Il'in scheme yields the exact solution and the remaining four schemes converge with $O(h^2)$ error.

In many practical computations, however, the Peclet number Pe is quite large and the mesh width $h$ cannot become infinitesimal. The results for the case when $\mathrm{Pe} \to \infty$ with the fixed mesh width $h$ are summarized in Table II.

It is noted that the upwind and the Samarskii schemes exhibit convergence to the exact solution for each mesh point. The maximum error of these schemes occurs at the mesh point closest to the outflow boundary, i.e. at $x = x_{N-1}$. At this point, the maximum error of the upwind scheme is $(T_1 - T_0)/(\mathrm{Pe}\,.\,h)$ whereas the maximum error of the Samarskii scheme is $2(T_1 - T_0)/(\mathrm{Pe}\,.\,h)^2$. Thus the errors of the Samarskii scheme are smaller than those of the

Table I. Error behaviour of $L_h^i$. Case 1: Pe fixed, $h \to 0$; $\theta = \mathrm{Pe}\,.\,h/2$

| Scheme | Value of $z$ | $\lim_{h\to 0} z^i$ | $\lim_{h\to 0} e_i$ |
|---|---|---|---|
| Upwind | $1+2\theta$ | $e^{\mathrm{Pe}\,.\,x_i}+O(\mathrm{Pe}^2\,.\,h)$ | $O(h)$ |
| Central | $\dfrac{1+\theta}{1-\theta}$ | $e^{\mathrm{Pe}\,.\,x_i}+O(\mathrm{Pe}^3\,.\,h^2)$ | $O(h^2)$ |
| Il'in | $e^{2\theta}$ | $e^{\mathrm{Pe}\,.\,x_i}$ | Zero |
| Samarskii | $1+2\theta+2\theta^2$ | $e^{\mathrm{Pe}\,.\,x_i}+O(\mathrm{Pe}^3\,.\,h^2)$ | $O(h^2)$ |
| Dennis | $\dfrac{2+2\theta+\theta^2}{2-2\theta+\theta^2}$ | $e^{\mathrm{Pe}\,.\,x_i}+O(\mathrm{Pe}^3\,.\,h^2)$ | $O(h^2)$ |
| Modified Samarskii | $\dfrac{1+\theta+\theta^2}{1-\theta+\theta^2}$ | $e^{\mathrm{Pe}\,.\,x_i}+O(\mathrm{Pe}^3\,.\,h^2)$ | $O(h^2)$ |

upwind scheme. The central difference scheme exhibits the familiar oscillatory property, the oscillations becoming unbounded when $N$ is even. The Il'in scheme, by design, is exact at each point. The surprising result is that the Dennis scheme and the modified Samarskii scheme provide solutions which do not converge to the exact solution anywhere in the interval (0, 1). The approximate solution in both cases converges to $T_0+(T_1-T_0)\,.\,x_i$, $1 \le i \le N-1$ whereas the exact solution $u(x_i) \to T_0$ as $\mathrm{Pe} \to \infty$ (see Figure 1). It is clear that these two difference schemes would be unsuitable for any practical computation (when Pe is even moderately large), even though both of these schemes have $O(h^2)$ truncation errors. Our numerical results, discussed in the next section, confirm these observations.

Gresho and Lee[5] have considered the values of diffusive flux at the outflow boundary for

Table II. Error behaviour of $L_h^i$. Case 2: $h$ fixed, $\mathrm{Pe} \to \infty$; $\theta = \mathrm{Pe}\,.\,h/2$

| Scheme | Value of $z$ | $\lim_{\mathrm{Pe}\to\infty} z^i$ | $\lim_{\mathrm{Pe}\to\infty} \left(\dfrac{u_i-T_0}{T_1-T_0}\right)$ | $\lim_{\mathrm{Pe}\to\infty} e_i$ |
|---|---|---|---|---|
| Upwind | $1+2\theta$ | $(2\theta)^i$ | $(2\theta)^{i-N}$ | $0,\ 1 \le i \le N-1$; max error $= e_{N-1} \cong \dfrac{T_1-T_0}{2\theta}$ |
| Central | $\dfrac{1+\theta}{1-\theta}$ | $(-1)^i(1+2i/\theta)$ | $-(i+\theta)/N$, $i$ odd; $i/N$, $i$ even | $-\infty$, $i$ odd; $(T_1-T_0)x_i$, $i$ even; ($N$ even) |
| | | | $(i+\theta)/(N+\theta)$, $i$ odd; $-i/(N+\theta)$, $i$ even | $(T_1-T_0)$, $i$ odd; $0$, $i$ even; ($N$ odd) |
| Il'in | $e^{2\theta}$ | $e^{2\theta i}$ | $(1-e^{2\theta i})/(1-e^{2\theta N})$ | $0,\ 1 \le i \le N-1$ |
| Samarskii | $1+2\theta+2\theta^2$ | $(2\theta^2)^i$ | $(2\theta^2)^{i-N}$ | $0,\ 1 \le i \le N-1$; max error $= e_{N-1} \cong \dfrac{(T_1-T_0)}{2\theta^2}$ |
| Dennis | $\dfrac{2+2\theta+\theta^2}{2-2\theta+\theta^2}$ | $1+4i/\theta$ | $i/N$ | $(T_1-T_0)x_i,\ 1 \le i \le N-1$ |
| Modified Samarskii | $\dfrac{1+\theta+\theta^2}{1-\theta+\theta^2}$ | $1+2i/\theta$ | $i/N$ | $(T_1-T_0)x_i,\ 1 \le i \le N-1$ |

the case when $\mathrm{Pe} \gg 1$. The diffusive flux at $x=1$ is given by

$$q \equiv -du/dx|_{x=1} = (T_1 - T_0) \, . \, \mathrm{Pe} \, . \, \frac{e^{\mathrm{Pe}}}{1-e^{\mathrm{Pe}}}$$

$$\rightarrow -(T_1 - T_0) \, . \, \mathrm{Pe}, \qquad \mathrm{Pe} \rightarrow \infty. \tag{8}$$

We have used the exact solution in equation (4) to obtain the above value of $q$. Gresho and Lee calculated the value of $q$ in the case of the finite difference schemes through the difference:

$$q \cong (u_{N-1} - u_N)/h. \tag{9}$$

Using the exact solution (6) of the difference schemes, we obtain

$$q \cong \frac{T_1 - T_0}{h} \, . \, z^{N-1} \, . \, \frac{(z-1)}{1-z^N}. \tag{10}$$

In Table III, we summarize the limiting behaviour of $q$ for the difference schemes $L_h^i$, $i = 1, \ldots, 6$.

It is noted that the limiting values of $q$ for $\mathrm{Pe} \gg 1$ do not approach the limiting values of $q$ given in equation (8) for *any* of the difference approximations. The closest values obtained are $-(T_1 - T_0) \, . \, N$ in the case of the upwind, Il'in and the Samarskii schemes. In the case of the central scheme, the limiting values of $q$ is $-(T_1 - T_0) \, . \, N$ for $N$ odd and $+\infty$ for $N$ even. The Dennis scheme and the modified Samarskii scheme yield $-(T_1 - T_0)$ as the limiting values.

Gresho and Lee[5] have given some emphasis to the flux calculations (9) and (10) obtained from the approximate solutions of various difference schemes. If one computes the value of $q \cong (u_{N-1} - u_N)/h$ from the exact solution (4) of the differential equation (3) one obtains

$$q \cong \frac{(T_1 - T_0)}{h} (1 - e^{-\mathrm{Pe}.h}) \, . \, \frac{e^{\mathrm{Pe}}}{1-e^{\mathrm{Pe}}}$$

$$\rightarrow -(T_1 - T_0) \, . \, N, \qquad \mathrm{Pe} \rightarrow \infty. \tag{11}$$

This shows that even the exact solution of the differential equation does not produce the correct value of diffusive flux when computed through equation (9) for $\mathrm{Pe} \gg 1$. It is thus unreasonable to expect *any* difference method to yield flux values that converge to

Table III. Behaviour of diffusive flux $q$. $h$ fixed, $\mathrm{Pe} \rightarrow \infty$, $\theta = \mathrm{Pe} \, . \, h/2$

| Scheme | Value of $z$ | $\lim_{\mathrm{Pe} \rightarrow \infty} q$ |
|---|---|---|
| Upwind | $1+2\theta$ | $-(T_1 - T_0) \, . \, N$ |
| Central | $(1+\theta)/(1-\theta)$ | $-(T_1 - T_0) \, . \, N \, . \, \frac{2(-1)^{N-1}}{1-(-1)^N}$ |
| Il'in | $e^{2\theta}$ | $-(T_1 - T_0) \, . \, N$ |
| Samarskii | $1+2\theta+2\theta^2$ | $-(T_1 - T_0) \, . \, N$ |
| Dennis | $(2+2\theta+\theta^2)/(2-2\theta+\theta^2)$ | $-(T_1 - T_0)$ |
| Modified Samarskii | $(1+\theta+\theta^2)/(1-\theta+\theta^2)$ | $-(T_1 - T_0)$ |

$-\text{Pe}\,.\,(T_1-T_0)$. The three difference schemes $L_h^1$, $L_h^3$ and $L_h^4$ yield the best limiting values that could reasonably be expected. However, if one were to use a graded mesh in the outflow boundary layer near $x=1$ such that $h<(\text{Pe})^{-1}$, then each difference scheme will produce correct values of the flux.

We conclude this section by giving a quick guide to detecting various properties of a difference scheme of the form (5) for solving the model problem (3). The solution of the difference equation is given by (6):

$$u_i = T_0 + (T_1 - T_0)\frac{(1-z^i)}{(1-z^N)}, \qquad 1 \le i \le N-1.$$

Here $z=b/a$ and $a$, $b$ are the coefficients of $u_{i+1}$ and $u_{i-1}$, respectively, in equation (5). For consistency, the difference scheme (5) must satisfy the condition

$$\frac{a-b}{a+b} = -\text{Pe}\,.\,h/2 + O(h^2); \qquad \text{Pe fixed, } h \text{ small}$$

i.e. $z=1+\text{Pe}\,.\,h+O(h^2)$, $h\to 0$.

When $h$ is fixed and $\text{Pe}\to\infty$, most consistent difference schemes give the limiting values of $z$ as 0, 1, $-1$ or $\pm\infty$. The behaviour of the solutions of such difference schemes is summarized in Table IV.

## 6. NUMERICAL RESULTS

We computed the numerical solutions of the one-dimensional convection diffusion equation (2) for several test functions. In each case, we inserted the test solutions in the differential equation to obtain the forcing term $f(x)$. The boundary values $u(a)$ and $u(b)$ were also obtained from the exact solution $u(x)$. We used the six finite difference schemes considered in the preceding sections with the value of $\lambda$ ranging from 1 to 100,000 and mesh width $h$ ranging from 0·1 to 0·005. First we present the results obtained with the model problem (equation (2) with $f=0$). The behaviour of the exact solution of the model problem is given in Figure 1. In Figure 2 we present the behaviour of the maximum errors of $L_h^i$ with increasing values of Pe for a typical mesh width $h=0{\cdot}01$. Here maximum error is defined as $\max_{1\le i\le N-1} |u_i - u(x_i)|$. The results were obtained on an IBM 4341 using double-precision arithmetic. The error curves for the Dennis scheme $L_h^5$ and the modified Samarskii scheme $L_h^6$ were found to be almost identical, especially for large Peclet numbers and we have only presented the error curve for $L_h^5$. The errors of the Il'in scheme for the model problem were found to be $O(10^{-15})$ for all values of $\text{Pe}\,.\,h$. This is the rounding error limit for the computer used.

From Figure 2 we observe that three schemes, viz. $L_h^2$ (Central), $L_h^4$ (Samarskii) and $L_h^5$ (Dennis), have comparable error behaviour when Pe is small. Each of these schemes has

Table IV. Behaviour of general difference schemes, $h$ fixed, $\text{Pe}\to\infty$

| Value of $z$ | Flux $q$ | Error $e_i$ | Solution behaviour |
|---|---|---|---|
| 0 | 0 | $T_1-T_0$ | Incorrect |
| 1 | $-(T_1-T_0)$ | $(T_1-T_0)\,.\,i/N$ | Incorrect |
| $-1$ | $-N(T_1-T_0)$, $N$ odd; $-\infty$, $N$ even | $\frac{1-(-1)^i}{1-(-1)^N}(T_1-T_0)$ | Oscillatory |
| $\pm\infty$ | $-N(T_1-T_0)$ | 0 | Correct |

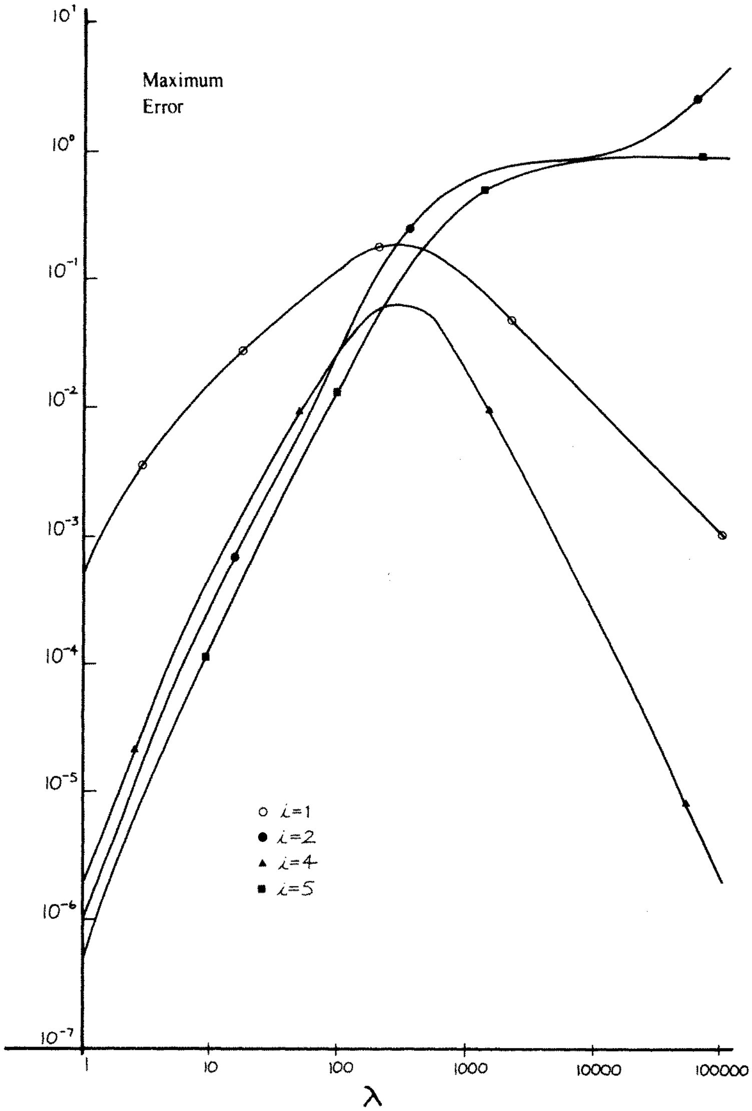


Figure 2. Maximum errors of the one-dimensional model problem: $i=1$ Upwind scheme; $i=2$ Central scheme; $i=4$ Samarskii scheme; $i=5$ Dennis scheme

$O(h^2)$ rate of convergence when Pe . $h$ is reduced. When Pe increases, the error behaviour of these three schemes is substantially changed. The central scheme exhibits oscillatory solutions for Pe . $h>2$ with rapidly increasing error when Pe is increased. The Dennis scheme as well as the modified Samarskii scheme have grossly incorrect, though smooth, solutions for large Pe. The only numerical solutions that have any resemblance to the exact solutions for large Peclet numbers are obtained with the upwind, Samarskii and Il'in schemes. We also observe from Figure 2 that each error curve exhibits a turning point usually for $\mathrm{Pe}\geq 2(h)^{-1}$. With a cruder mesh these turning points are observed for smaller values of Pe. Of course, with finer mesh these points could be pushed further.

The behaviour of the Dennis scheme $L_h^5$ is clearly exhibited in Figure 3 where we present

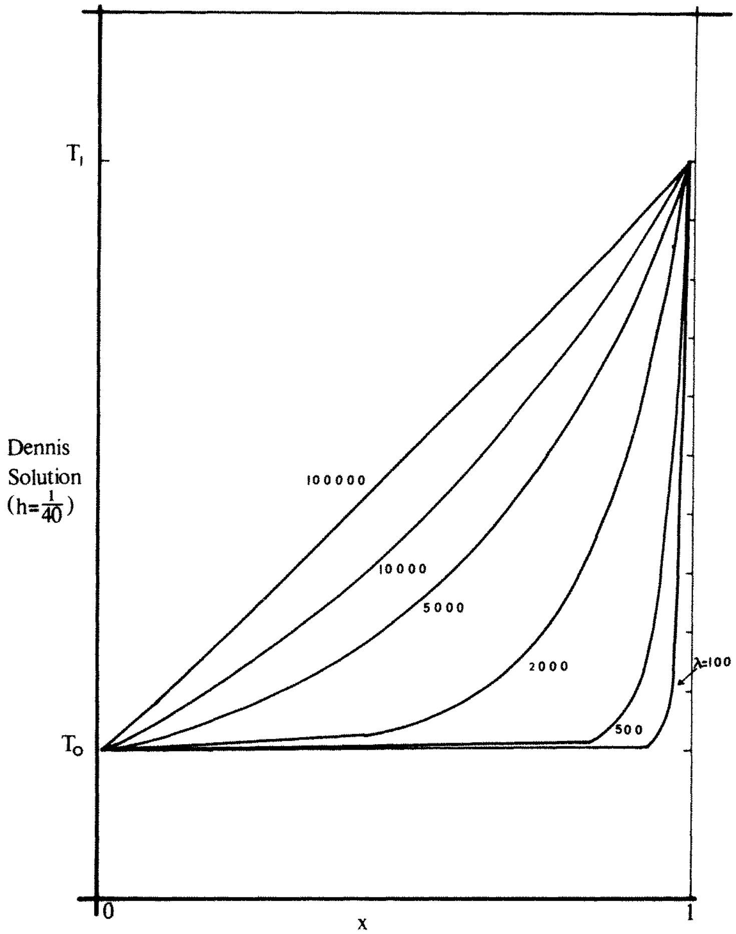


Figure 3. Behaviour of the Dennis scheme for large Peclet numbers (one-dimensional model problem)

the solution profiles of the model problem for a typical mesh ($h=0{\cdot}025$). As Pe increases from 1 to 100, the behaviour of the solution curves remains consistent with the exact solutions in Figure 1. As Pe increases beyond 100, these solution curves no longer approach the exact values; instead these curves reverse the previous trend and begin approaching the straight line $u(x)=T_0+(T_1-T_0)x$. Such turning points are observed at Pe = 200, $h=0{\cdot}01$ and Pe = 50, $h=0{\cdot}05$. At Pe = $10^5$, $h=0{\cdot}025$ the solution curve of the Dennis scheme $L_h^5$ has come very close to the straight line (Figure 3).

We also computed the numerical solutions of equation (2) using the six difference schemes for several other test problems with $f\neq 0$. The difference schemes $L_h^i$, $i=2,3,4,5,6$ exhibited $O(h^2)$ convergence when $\lambda$ was small. The upwind scheme exhibited $O(h)$ convergence, as expected. As $\lambda$ was increased, the Il'in ($i=3$) and the Samarskii ($i=4$) schemes behaved much like the upwind scheme ($i=1$) whereas the Dennis ($i=5$) and the modified Samarskii ($i=6$) schemes yielded grossly inaccurate solutions. The central scheme ($i=2$) yielded oscillatory solutions, as expected.

Kellogg and Tsan[10] have proved discretization error estimates of order $h^2$ when $h<\lambda^{-1}$ for both the Il'in and the Samarskii schemes. For large values of $\lambda$, both of these schemes suffer a loss in the order of accuracy. This loss in the order of accuracy is typical of all three-point approximations (of positive type) of equation (2).[10]

In the next section we present some results for the two-dimensional problem of equation (1).

## 7. TWO-DIMENSIONAL CONVECTION–DIFFUSION EQUATION

We now consider the two-dimensional convection–diffusion equation (1). The analysis in this case is complicated but the results of one dimension are expected to hold in higher dimensions.

We computed the numerical solutions of equation (1) in a unit square $0 \le x, y \le 1$ for a variety of test problems. The forcing term $f(x, y)$ and the boundary values of $u(x, y)$ were, in each case, obtained from the known exact solutions. The values of maximum errors with $L_h^i$ $(i = 1, 2, 3, 4, 5)$ for a typical test problem are presented in Table V. Here the exact solution is $u(x, y) = 2x(x-1)(\cos 2\pi y - 1)$ and a uniform mesh width with $h = 0{\cdot}05$ is used to cover the unit square. The maximum error is defined as $\max_{1 \le k, l \le N-1} |u_{k,l} - u(x_k, y_l)|$. The results in Table V are obtained on an IBM 4341 using single-precision arithmetic.

We observe from Table V that the schemes $L_h^i$ $(i = 2, 3, 4, 5)$ yield comparable solutions when the values of $\lambda_1, \lambda_2$ are small. When the values of $\lambda_j$ are increased, the Il'in and the Samarskii schemes start behaving exactly like the upwind scheme. The Dennis scheme $(i = 5)$ begins yielding grossly inaccurate solutions for large values of $\lambda_j$, its error reaches the 100 per cent level as $\lambda_j$ are increased. The central scheme $(i = 2)$ yields oscillatory solutions for $\lambda_j > 100$ $(h = 0{\cdot}05)$.

In the last column of Table V we have also presented the errors obtained with a fourth-order difference scheme which has been designed especially for the two-dimensional convection–diffusion equation (1). This scheme remains stable for all values of $\lambda_j$ and yields highly accurate solutions for the whole range of $\lambda_j$. It has been derived as a generalization of a nine-point fourth-order discretization of the Poisson equation. For details, see Reference 6.

We have also developed a fourth-order difference scheme for equation (1) when the

Table V. Maximum errors for the two-dimensional convection–diffusion equation. Mesh width $h = 0{\cdot}05$. Exact solution $u(x, y) = 2x(x-1)(\cos 2\pi y - 1)$

| $\lambda_1$ | $\lambda_2$ | Upwind $i=1$ | Central $i=2$ | Il'in $i=3$ | Samarskii $i=4$ | Denis $i=5$ | Fourth-order scheme |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0·0197 | 0·4691(−2)* | 0·4492(−2) | 0·4142(−2) | 0·4386(−2) | 0·3285(−3) |
| 10 | 1 | 0·0823 | 0·3733(−2) | 0·4264(−2) | 0·1305(−1) | 0·7194(−2) | 0·2135(−3) |
| 10 | 10 | 0·1266 | 0·1009(−1) | 0·1132(−1) | 0·2558(−1) | 0·1621(−1) | 0·1458(−3) |
| 100 | 10 | 0·1797 | 0·4194(−2) | 0·1062 | 0·1207 | 0·1929 | 0·1018(−3) |
| 100 | 100 | 0·1825 | 0·1929(−1) | 0·1176 | 0·1350 | 0·2216 | 0·1679(−2) |
| 500 | 100 | 0·2174 | 0·1174(−1) | 0·1952 | 0·2002 | 0·6744 | 0·8767(−3) |
| 1,000 | 100 | 0·2077 | 0·6894(−2) | 0·1951 | 0·1978 | 0·8691 | 0·4563(−3) |
| 2,000 | 100 | 0·1988 | 0·4940(−2) | 0·1921 | 0·1935 | 0·9628 | 0·2724(−3) |
| 5,000 | 100 | 0·1940 | 0·4015(−2) | 0·1912 | 0·1917 | 0·9918 | 0·7367(−4) |
| 100,000 | 10,000 | 0·2143 | 0·1121(−1) | 0·2142 | 0·2142 | 1·0000 | 0·3924(−3) |

* $0{\cdot}4691(-2) = 0{\cdot}4691 \times 10^{-2}$

coefficients $\lambda_1$, $\lambda_2$ are functions of $x$, $y$. This scheme is suitable for solving the Navier–Stokes equations. Details of this scheme are being published elsewhere.[7]

We note here that the Dennis scheme $L_h^5$ was developed by Dennis *et al.*[4] for solving three-dimensional Navier–Stokes equations. Dennis *et al.*[4] designed their scheme to obtain diagonal dominance and to solve the problem of flow of a viscous incompressible fluid in a three-dimensional cavity. While we have not carried out any computations in three dimensions, there is evidence[1,12] that the Dennis scheme suffers from large artificial diffusion at high Reynolds numbers. Agarwal[1] has presented three-dimensional results showing the solutions of Dennis *et al.*[4] to be highly inaccurate for moderate to large Reynolds numbers.

We used the Il'in scheme $L_h^3$ to solve the two-dimensional problem of a viscous incompressible fluid flow in a driven cavity. The results for smaller Reynolds numbers (up to 100, $h = 0{\cdot}05$) were found to be comparable to the central scheme solutions. For large Reynolds numbers, we encountered numerical instabilities and the solutions were no better than the upwind scheme solutions.

## 8. CONCLUSIONS

The major conclusion of this paper is that there is no universal 'second-order' scheme of the type (5) for the one-dimensional problem. When the mesh width $h$ is fixed, and the Peclet number is increased one expects to see either (i) the correct though not very accurate behaviour of the upwind scheme, or (ii) the oscillatory and possibly inaccurate behaviour of the central scheme, or (iii) the smooth though grossly inaccurate behaviour of the Dennis scheme. It seems that for the one-dimensional convection–diffusion equation there is no other limiting behaviour in general.

We have clearly illustrated the pitfalls of deriving finite difference schemes with assumptions such as $e^\theta \cong 1+\theta+\theta^2$ and $(1+\theta)^{-1} \cong 1-\theta+\theta^2$. We have shown that the difference schemes thus obtained are satisfactory only when the above assumptions are valid, i.e. when $\theta$ is small. When one uses such difference schemes for large values of $\theta$, one obtains highly inaccurate solutions as has been conclusively demonstrated in case of the Dennis scheme and the modified Samarskii scheme.

We have also presented evidence to demonstrate that the behaviour of the one-dimensional difference schemes carries over to the higher dimensions. It is, however, possible to derive new difference schemes in higher dimensions which are highly accurate, stable as well as cost effective. Such schemes may not have lower-dimensional counterparts though.

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